# Tutorial 6

In this tutorial, we will focus on finding geodesics on some simple surfaces.

We first recall the def. of geodesic on a surface:

$C(s)$ = a regular curve lying in an embedded surface $\Sigma$ in $\mathbb{R}^3$, $s$ = arc-length.

$$c''(s) = \underset{(\in T_{c(s)}\Sigma)}{(c''(s))^T} + \underset{(\in N_{c(s)}\Sigma)}{(c''(s))^N}$$

We call $c(s)$ is a geodesic if $(c''(s))^T \equiv 0 \ \forall s$.

$\left( k_g(s) \triangleq \| (c''(s))^T \| \right.$, geodesic curvature, i.e. $\left. k_g \equiv 0. \right)$

$X(u^1, u^2)$ = a coordinate patch of $\Sigma$

$$c(s) = X(u^1(s), u^2(s))$$

$$c' = X_i \frac{du^i}{ds}$$

$$c'' = X_i \frac{d^2u^i}{ds^2} + X_{ij} \frac{du^i}{ds} \frac{du^j}{ds}$$

Recall $X_{ij} = \Gamma_{ij}^k X_k + h_{ij} \mathbb{U}$, $\mathbb{U} = \dfrac{X_1 \times X_2}{|X_1 \times X_2|}$.

$$\Rightarrow \quad c'' = X_k \frac{d^2 u^k}{ds^2} + \Gamma^k_{ij} \frac{du^i}{ds}\frac{du^j}{ds} X_k$$

$$+ h_{ij} \frac{du^i}{ds}\frac{du^j}{ds} U$$

$$\Rightarrow \quad (c'')^T = \left( \frac{d^2 u^k}{ds^2} + \Gamma^k_{ij} \frac{du^i}{ds}\frac{du^j}{ds} \right) X_k$$

$$\Rightarrow \quad \text{geodesic} \iff \frac{d^2 u^k}{ds^2} + \Gamma^k_{ij} \frac{du^i}{ds}\frac{du^j}{ds} = 0 \quad (*), \quad \forall k = 1, 2.$$

Geodesic equation

a 2nd order non-linear ODE system

By ODE theory, given initial data : $(u^1(0), u^2(0))$

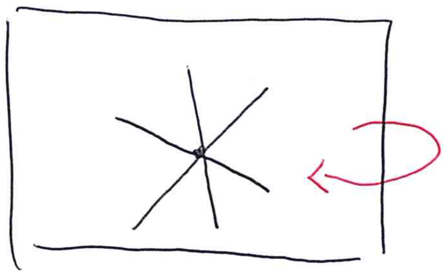
$$\left( \frac{d}{ds}\Big|_{s=0} u^1, \frac{d}{ds}\Big|_{s=0} u^2 \right)$$

One can solve $(u^1(s), u^2(s))$ locally i.e. $\exists\, \varepsilon > 0$ s.t. $(u^1(s), u^2(s))$ s.f. $(*)$ for $s \in [0, \varepsilon)$.

① Geodesics on a circular cylinder : One way to find them is to solve the geodesic equation ; another way :

circular cylinder $\overset{\text{Isom.}}{\cong}$ flat plane geodesics in a flat plane = straight lines

geodesic is a concept generalizing the straight line in $\mathbb{R}^n$ when you considering some curved surface (manifold).



$$X(\theta, z) = (\cos\theta, \sin\theta, z), \quad \theta \in [0, 2\pi), \ z \in \mathbb{R}$$

$(\theta, z) = (x^1, x^2)$

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

$$= 0, \quad \forall\, i, j, k = 1, 2.$$

$\Rightarrow$ The geodesic equation is

$$\begin{cases} \dfrac{d^2\theta}{ds^2} = 0 \\ \dfrac{d^2 z}{ds^2} = 0 \end{cases}$$

Initial data: $p = (1, 0, 0)$, $V_p = c_1 \frac{\partial}{\partial \theta} + c_2 \frac{\partial}{\partial z}$

$$\Rightarrow \begin{cases} \theta(s) = as + b \\ z(s) = cs + d \end{cases}$$

$$X(\theta(0), z(0)) = (1, 0, 0)$$

$$\Rightarrow \theta(0) = \underbrace{\cancel{0}, \quad z(0) = 0}_{2k\pi,\ \text{for some } k \in \mathbb{Z}}$$

$$\left.\frac{d}{ds}\right|_{s=0} X(\theta(s), z(s)) = c_1 X_\theta + c_2 X_z$$

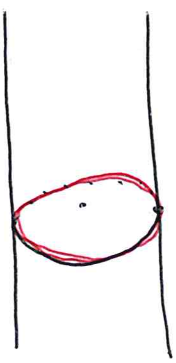
$$\parallel$$

$$X_\theta \cdot \theta'(0) + X_z z'(0)$$

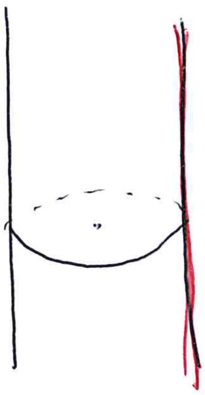
$$\rightarrow \theta'(0) = c_1, \quad z'(0) = c_2$$

$$\Rightarrow \begin{cases} \theta(s) = c_1 s + 2k\pi \\ z(s) = c_2 s \end{cases} \quad \text{for some } k \in \mathbb{Z}$$

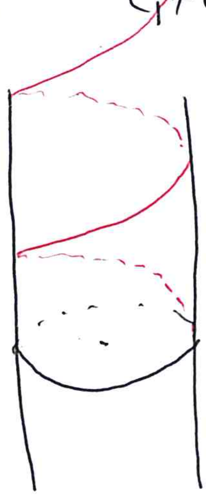
(1) $c_2 = 0$



(2) $c_1 = 0$



(3) $c_1 \neq 0, c_2 \neq 0$



helix

② Geodesics on the standard sphere $S^2$

$$X(u,v) = (\cos u \cos v, \ \sin u \cos v, \ \sin v)$$

$X(u_0, v)$ = longitude, $u_0$ fixed, $|X_v| \equiv 1$.

$$X_v = (-\cos u \sin v, \ -\sin u \sin v, \ \cos v)$$

$$X_{vv} = (-\cos u \cos v, \ -\sin u \cos v, \ -\sin v)$$

$$X_u = (-\sin u \cos v, \ \cos u \cos v, \ 0)$$

$$\langle X_{vv}, X_v \rangle = \cos^2 u \sin v \cos v + \sin^2 u \sin v \cos v - \sin v \cos v$$

$$= 0$$

$$\langle X_{vv}, X_u \rangle = 0$$

$\Rightarrow (X_{vv})^T = 0 \Rightarrow$ longitudes are geodesics!

Symmetry $\Rightarrow$ great circles are geodesics!

$\quad\quad\quad\quad\quad\quad \|$

$S^2 \cap$ Plane passing through center of $S^2$.

Conversely, if

$\alpha(s)$ = geodesic in $S^2$ parametrized by arc-length

$$\alpha'' = (\alpha'')^T + (\alpha'')^N$$

$$= (\alpha'')^T + \langle \alpha'', U \rangle U$$

geodesic $\Rightarrow$ $\alpha'' = \langle \alpha'', U \rangle U$

$U = \alpha(t)$ at $\alpha(t)$

$$\Rightarrow \alpha'' = \langle \alpha'', \alpha \rangle \alpha$$

$$(U \times T)' = (\alpha \times \alpha')'$$

$$= \cancelto{0}{\alpha' \times \alpha'} + \cancelto{0}{\alpha \times \alpha''}$$

$$= 0$$

$\tilde{N} := U \times T$ = a constant tangent vector s.t

$\langle \tilde{N}, \alpha \rangle = \langle \alpha \times \alpha', \alpha \rangle = 0$ i.e

$\alpha$ belongs to the plane with normal $\tilde{N}$

ALSO NOTE : $\langle U, \tilde{N} \rangle = 0$ i.e $U$ belongs to the plane

i.e the plane contains the center of $S^2$

$\Rightarrow \alpha =$ great circle.

$\square$

(3) Geodesics on a circular cone:

$$X(u,v) = (u\cos v, u\sin v, u)$$

$$X_u = (\cos v, \sin v, 1) \qquad (g_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & u^2 \end{pmatrix}$$

$$X_v = (-u\sin v, u\cos v, 0)$$

We will go on studying the geodesics on a circular cone next time